HOMOGENIZATION OF A DOUBLE POROSITY MODEL IN DEFORMABLE MEDIA

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Abstract. The paper addresses the homogenization of a family of micro-models for the flow of a slightly compressible fluid in a poroelastic matrix containing periodically distibuted poroelastic inclusions, with low permeabilities and with imperfect contact on the interface. The micro-models are based on Biot's system for consolidation processes in each phase, with interfacial barrier formulation. Using the two-scale convergence technique, it is shown that the derived system is a general model of that proposed by Aifantis, plus an extra memory term.

1. Introduction

The interaction between fluid flow and solid deformation in porous media is of great importance in petroleum engineering and geomechanics, biosciences, chemical processes and many industrial applications [12, 11, 20].

Some type of porous rocks, like aquifers and petroleum reservoir systems, may contain fractures. It is known that flows in such media occur mainly in the fracture region and the dominant fluid storage is in the matrix blocks. In this situation, reservoir possesses two porous structures, one related to the matrix, and the other related to fractures. This notion of double porosity/permeability has first been introduced by Barenblatt, Zheltov and Kochina [7] to model the flow of a slightly compressible fluid within naturally fractured porous media. The proposed model is a system of two partial differential equations in a two-medium description, with Darcy's law in each phase, plus exchange extra-terms representing the interfacial coupling that results from the interaction, at the micro-scale, between the two phases, see (1.6)-(1.7) below. This was derived under the main assumption that the fluid pressure is uniformly distributed at the surface of each phase.

Generally, fractured rock formations present at the micro-scale high degrees of heterogeneity and permeability is mainly determined by the size of the pores and the connectedness of the fractures system. So any mathematical modeling of fluid flow in such porous media must take into account the rapid spatial variation of the phenomenological parameters. Furthermore, from the numerical point of view, modeling of such systems at the local scale yields a huge number of discretized equations, so computations will be fastidious and intractable. To deal with such highly heterogeneous domains, the idea is to replace the medium by an effective one. Homogenization techniques, like two-scale convergence method, have been used to rigorously derive an effective double-porosity model for the Barenblatt, Zheltov and Kochina (BZK) system, see for instance H. Ene and D. Polisevski

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[14]. However, this model does not take into account the elastic behavior of the solid. In fact, a rise in pore pressure of the fluid produces a dilation of the solid mass. On the other hand, compression of the medium will increase pore pressure. This coupled pressure-deformation was first introduced by Terzaghi [19] in the one-dimensional setting and gave the first soil consolidation problem for a homogeneous elastic porous medium. Later, M. A. Biot [8] has developed in the multidimensional setting a linear theoretical analysis for the behavior of a fluid saturated poroelastic medium. The model was based on macroscopic description of the phenomenological and physical quantities where the representative volume element is described as the superposition of a particle of fluid and a particle of solid. Assuming that microstructures are periodically distributed and that the pore scale is very small compared to the macroscopic scale, a two-scale asymptotic expansion technique can be used to rigorously justify this Biot's model. The microscopic models are based on the linear elasticity equations in the skeleton and on the Stokes equations in the fluid with appropriate transmission conditions. For more details, we refer the reader to the earlier work by Auriault and Sanchez-Palencia [6].

Because of the coupling between the deformation and fluid pressure in double porosity rocks, which must be understood in order to predict reservoir or aquifer behavior, the concept of double porosity has been developed by E.C. Aifantis [1] to model oil flow in porous elastic rocks. More precisely, E. C. Aifantis gave a phenomenological model for flow of a weakly compressible fluid in a complex and heterogeneous medium where a system of partial differential equations is given and generalizing Biot's consolidation model by taking into account the basic physics of flow through fractured media with interscale couplings. The proposed model reads as follows:

$$
\begin{aligned}
-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u}) + \alpha_1\nabla p_1 + \alpha_2\nabla p_2 &= \mathbf{f}, && (1.1)\\
c_1\partial_t p_1 + \alpha_1 \mathrm{div}\,(\partial_t\mathbf{u}) - K_1\Delta p_1 + g(p_1-p_2) &= h_1, && (1.2)\\
c_1\partial_t p_2 + \alpha_2 \mathrm{div}\,(\partial_t\mathbf{u}) - K_2\Delta p_2 - g(p_1-p_2) &= h_2 && (1.3)
\end{aligned}
$$

where u is the displacement of the medium; λ and μ are the dilation and shear moduli of elasticity, respectively; p_i is the pressure of the fluid in phase (i); c_i the compressibility, K_i the permeability and α_i is the Biot-Willis parameters [9]. We note that if we let the volume of fissures shrinks to zero so that c_2, α_2, K_2, g become negligible then the system (1.1)-(1.3) reduces to the classical Biot system with single porosity [8]:

$$
\begin{aligned}
-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\mathrm{div}\mathbf{u}) + \alpha_1\nabla p_1 &= \mathbf{f}, && (1.4)\\
c_1\partial_t p_1 + \alpha_1 \mathrm{div}\,(\partial_t\mathbf{u}) - K_1\Delta p_1 &= h_1. && (1.5)
\end{aligned}
$$

On the other hand, by neglecting the deformation effects λ, μ and α_i the system (1.1)-(1.3) reduces to the BZK model [7]:

$$
\begin{aligned}
c_1\partial_t p_1 - K_1\Delta p_1 + g\,(p_1-p_2) &= h_1, && (1.6)\\
c_2\partial_t p_2 - K_2\Delta p_2 - g\,(p_1-p_2) &= h_2 && (1.7)
\end{aligned}
$$

Aifantis' theory of consolidation with the concept of double porosity unify then the proposed models (1.4)-(1.5) of Biot for consolidation of deformable porous media with single porosity and (1.6)-(1.7) of BZK model for fluid flow through undeformable porous media with double porosity. Note also that a mathematical

justification of the Aifantis model has been established in [2]. More precisely, it is considered micro-models with periodically distributed poroelastic inclusions, embedded in an extra poroelastic matrix, with imperfect contact on the interface. The micro-model is based on Biot's system for consolidation processes with interfacial barrier formulation. The macro-model is then derived by means of the two-scale convergence technique and it reads as follows:

$$-\mathrm{div}\sigma\left(\mathbf{u}\right)+\alpha_1\nabla p_1+\alpha_2\nabla p_2=\mathbf{f}, \tag{1.8}$$

$$\partial_t\left(\widetilde{c_1}p_1+\beta_1:e\left(\mathbf{u}\right)\right)-\mathrm{div}\left(K_1\nabla p_1\right)+\widetilde{g}\left(p_1-p_2\right)=h_1 \tag{1.9}$$

$$\partial_t\left(\widetilde{c_2}p_2+\beta_2:e\left(\mathbf{u}\right)\right)-\mathrm{div}\left(K_2\nabla p_2\right)-\widetilde{g}\left(p_1-p_2\right)=h_2 \tag{1.10}$$

where σ, α_i, β_i and K_i are some effective tensors, $i = 1, 2$. See [2] for more details. It is then worth pointing out that the Aifantis model (1.1)-(1.3) can be seen as a special case of the homogenized model (1.8)-(1.10) ($\beta_i = \alpha_i = \gamma_i \mathrm{I}_3$, γ_i being a scalar and I_3 the identity matrix).

In this paper, we consider a family of microscopic models for the fluid flow in a periodic poroelastic medium made of two constituents : the matrix and the inclusions, where the material properties change rapidly on a small scale characterized by a parameter ε representing the periodicity of the medium. We shall make the essential assumption that these inclusions have sizes large enough compared with the sizes of pores so that it makes sense to consider these media as poroelastic materials.

An interesting question is to investigate the limiting behavior of such media when the flow in the inclusions presents very high frequency spatial variations as a result of a relatively very low permeability when comparing to the matrix permeability, since pore flow velocities in the porous matrix can be high compared to movement through the interconnected pore spaces in the inclusions. This leads especially to rescale the flow potential in the inclusions by ε^2, as in Arbogast & *al.* [5]. The main objective of this paper is to derive a general model from the point of view of homogenization theory. It will be seen that the macro-model is in some sense the limit of a family of periodic micro-models in which the size of the periodicity approach zero. It is shown that the overall behavior of fluid flow in such heterogeneous media with low permeability at the micro-scale may present memory terms. It is also shown that in such anisotropic media, with different coupling interaction properties in different directions, the Biot-Willis parameters are, as in [2], matrices and no longer scalars, as it is usually considered in the poroelasticity litterature, since it is assumed there that the medium is homogeneous and isotropic.

The paper is organized as follows. In the next section 2, we give the geometrical setting, the family of the periodic micro-models, and state the main result of the paper. Section 3 is devoted to the proof of the main result with the help of the two-scale convergence technique. Finally, in section 4, we conclude the paper with some remarks.

2. Setting of the micro-model and main result

The aim of this section is to provide a detailed set up of the studied microstructure problem, introduce some necessary notations, basic mathematical tools as well as the notion of two-scale convergence, auxiliary problems, and then formulate the main result of the paper.

We consider Ω a bounded and smooth domain in $\mathbb{R}^3$, $\varepsilon > 0$ a sufficiently small parameter ($\varepsilon \ll 1$) and $Y =]0,1[^3$ the generic cell of periodicity. We assume that Y is divided as $Y = Y_1 \cup Y_2 \cup \Gamma$ where Y_1, Y_2 are two connected open subsets of Y and Γ is a smooth surface that separates them. They are such that $\overline{Y_2} \subset Y$, $Y_1 \cap Y_2 = \emptyset$, $\Gamma = \overline{Y_1} \cap \overline{Y_2} = \partial Y_2$ and $\partial Y_1 = \Gamma \cup \partial Y$. We denote $\mathbf{n} = (n_i)_{1 \leq i \leq 3}$ the unit normal vector on Γ pointing outward with respect to Y_1. Let χ_1, χ_2 denote respectively the characteristic function of Y_1, Y_2 extended by Y-periodicity to $\mathbb{R}^3$. Denote for $x \in \Omega$, $\chi_i^\varepsilon(x) = \chi_i(x/\varepsilon)$ and set

$$\Omega_i^\varepsilon = \{x \in \Omega : \chi_i^\varepsilon(x) = 1\} \text{ and } \Gamma^\varepsilon = \overline{\Omega_1^\varepsilon} \cap \overline{\Omega_2^\varepsilon}.$$

Let $Z_i = \cup_{e \in \mathbb{Z}^3} (Y_i + e)$, $i = 1, 2$. As in [3], we shall assume that the subset Z_1 is smooth and connected open subset of $\mathbb{R}^3$.

With the above assumptions, the material occupying the domain Ω_2^ε is then embedded in the material occupying Ω_1^ε, and the interface Γ^ε is the boundary $\partial\Omega_2^\varepsilon$. We observe that the boundary of Ω_1^ε consists of two parts the outer boundary $\partial\Omega$ and Γ^ε. Usually, the material Ω_1^ε is referred to the matrix material while the material Ω_2^ε to the inclusions. Note that no connectedness assumption is made on the material part Ω_2^ε.

Let $T > 0$ and $t \in [0, T]$ denote the time variable. We set the space-time domains $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \Gamma$, $Q_i^\varepsilon = (0, T) \times \Omega_i^\varepsilon$, and $\Sigma^\varepsilon = (0, T) \times \Gamma^\varepsilon$.

Let us assume that each phase $(\Omega_1^\varepsilon, \Omega_2^\varepsilon)$ is occupied by a porous and deformable material through which a slightly compressible and viscous fluid flow diffuses. Let $\mathbf{u}_i^\varepsilon$ denote the displacement of the medium Ω_i^ε, $i = 1, 2$. The equation of motion in $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ is given by

$$- \operatorname{div}\sigma_1^\varepsilon = \mathbf{f}_1, \text{ in } \Omega_1^\varepsilon, \tag{2.1}$$

$$-\operatorname{div}\sigma_2^\varepsilon = \mathbf{f}_2, \text{ in } \Omega_2^\varepsilon \tag{2.2}$$

where σ_i^ε is the stress tensor which satisfies a constitutive equation of linear poroelasticity of the form [12]:

$$\sigma_i^\varepsilon = \mathbb{A}_i^\varepsilon \mathrm{e}\left(\mathbf{u}_1^\varepsilon\right) - \alpha_i^\varepsilon p_i^\varepsilon \mathrm{I}_3, \text{ in } \Omega_i^\varepsilon \tag{2.3}$$

and $\mathbf{f}_i \in L^2(\Omega)^3$ is the volume distributed force in the corresponding medium, $i = 1, 2$. It is assumed that $\mathbf{f}_i$ is independent of ε. In (2.3), $\mathbb{A}_1^\varepsilon$ and $\mathbb{A}_2^\varepsilon$ are fourth rank elasticity tensors, $\mathrm{e}(\cdot)$ is the linearized strain tensor, I_3 is the identity matrix, p_i^ε is the pressure and α_i^ε is the Biot-Willis parameter in the poroelastic material Ω_i^ε [9].

Let c_1^ε (resp. c_2^ε) and K_1^ε (resp. K_2^ε) denote respectively the porosity and the permeability of the medium Ω_1^ε (resp. Ω_2^ε). The equation for mass conservation in each phase reads as follows:

$$\partial_t \left(c_1^\varepsilon p_1^\varepsilon + \alpha_1^\varepsilon \operatorname{div}\mathbf{u}_1^\varepsilon\right) - \operatorname{div}\left(K_1^\varepsilon \nabla p_1^\varepsilon\right) = 0, \text{ in } \Omega_1^\varepsilon, \tag{2.4}$$

$$\partial_t \left(c_2^\varepsilon p_2^\varepsilon + \alpha_2^\varepsilon \operatorname{div}\mathbf{u}_2^\varepsilon\right) - \operatorname{div}\left(K_2^\varepsilon \nabla p_2^\varepsilon\right) = 0 \text{ in } \Omega_2^\varepsilon. \tag{2.5}$$

On the interface Γ^ε, we associate to (2.1)-(2.2) the following transmission conditions:

$$\mathbf{u}_1^\varepsilon = \mathbf{u}_2^\varepsilon,\ \sigma_1^\varepsilon \cdot \mathbf{n}^\varepsilon = \sigma_2^\varepsilon \cdot \mathbf{n}^\varepsilon \tag{2.6}$$

and to (2.4)-(2.5) the well-known open-pore conditions:

$$\left(K_1^\varepsilon \nabla p_1^\varepsilon\right) \cdot \mathbf{n}^\varepsilon = \left(K_2^\varepsilon \nabla p_2^\varepsilon\right) \cdot \mathbf{n}^\varepsilon,\ \left(K_1^\varepsilon \nabla p_1^\varepsilon\right) \cdot \mathbf{n}^\varepsilon = -g^\varepsilon \left(p_1^\varepsilon - p_2^\varepsilon\right). \tag{2.7}$$

where $\mathbf{n}^\varepsilon$ stands for the unit normal vector on Γ^ε pointing outward with respect to Ω_1^ε, and g^ε is the hydraulic permeability of the thin layer Γ^ε. The interface conditions (2.7) are also known as *Deresiewicz-Skalak boundary conditions* [13]. Taking the limit on the thickness of the thin layer, one can prove that these conditions are the only ones that are fully consistent with the validity of the poroelasticity's equations throughout heterogeneous media presenting interfaces across which the diffusion is discontinuous, see [15]. Observe that when $g^\varepsilon = \infty$, (2.7) reduces to the standard transmission condition, that is a perfect hydraulic contact on the interface, and when $g^\varepsilon = 0$, condition (2.7) implies no motion of the fluid relative to the solid. Here, in this paper we shall assume that none of these conditions are fulfilled. See assumption (H4) below.

On the exterior boundary $\partial\Omega\backslash\Gamma^\varepsilon$, we assume the homogeneous Dirichlet boundary condition:

$$\mathbf{u}_1^\varepsilon = \mathbf{0} \text{ and } p_1^\varepsilon = 0. \tag{2.8}$$

Finally, the system (2.4)-(2.8) is supplemented by the following initial conditions:

$$\mathbf{u}_1^\varepsilon(0,\cdot) = \mathbf{0},\ p_1^\varepsilon(0,\cdot) = 0 \text{ in } \Omega_1^\varepsilon, \tag{2.9}$$
$$\mathbf{u}_2^\varepsilon(0,\cdot) = \mathbf{0},\ p_2^\varepsilon(0,\cdot) = 0 \text{ in } \Omega_2^\varepsilon. \tag{2.10}$$

To deal with periodic homogenization with microstructures, we shall assume the followings:

(H1) There exists Y-periodic, fourth rank tensor-valued functions $\mathbb{A}_i(y)$, $i = 1,2$ and continuous on $\mathbb{R}^3$ such that
$$\mathbb{A}_i^\varepsilon(x) = \mathbb{A}_i\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega,$$
and
$$(\mathbb{A}_i(y)\,\Xi : \Xi) \geq C\,(\Xi : \Xi)\,.$$
for all $y \in \mathbb{R}^3$ and $\Xi \in \mathcal{M}^{3\times3}_{\mathrm{sym}}(\mathbb{R})$;

(H2) There exist Y-periodic real-valued functions $c_i(y)$, $i = 1,2$ and continuous on $\mathbb{R}^3$ such that
$$c_i^\varepsilon(x) = c_i\left(\frac{x}{\varepsilon}\right),\ x \in \Omega$$
and $c_i(y) \geq C > 0$ for all $y \in \mathbb{R}^3$;

(H3) There exist Y-periodic matrix-valued functions $K_i(y)$, $i = 1,2$, continuous on $\mathbb{R}^3$ such that
$$K_1^\varepsilon(x) = K_1\left(\frac{x}{\varepsilon}\right), \quad K_2^\varepsilon(x) = \varepsilon^2 K_2\left(\frac{x}{\varepsilon}\right), \quad x \in \Omega \tag{2.11}$$
and
$$\langle K_i\xi, \xi\rangle \geq C\,|\xi|^2\,,\ i = 1,2$$
for all $y \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$;

(H4) There exists a function $g \in \mathcal{C}\left(\mathbb{R}^3\right)$, Y-periodic such that
$$g^\varepsilon(x) = \varepsilon g(x/\varepsilon)\,, \quad x \in \mathbb{R}^3 \text{ and } \inf_{y\in\mathbb{R}^3} g(y) \geq C > 0.$$

(H5) The Biot-Willis parameter α_i^ε is defined a.e. in Ω as follows:
$$\alpha_1^\varepsilon(x) = \alpha_1 \text{ for } x \in \Omega_1^\varepsilon \text{ and } \alpha_2^\varepsilon(x) = \varepsilon\alpha_2 \text{ for } x \in \Omega_2^\varepsilon \tag{2.12}$$
where α_i is a positive constant, $i = 1,2$.

Here and thoughout this paper, the quantity C denotes various positive constants independent of $\varepsilon > 0$, of the subscript $i = 1,2$ and the microscopic variable $y \in \mathbb{R}^3$.

Remark 2.1. *We have chosen a particular scaling of the permeability coefficients in (2.11). This means that the permeability is much larger in the network of pores of the inclusions than in the porous matrix rocks. This gives that both terms* $\int_{\Omega_1^\varepsilon} |\nabla p_1^\varepsilon|^2 \, dx$ *and* $\varepsilon^2 \int_{\Omega_2^\varepsilon} |\nabla p_2^\varepsilon|^2 \, dx$ *have the same order of magnitude and thus leading to a balance in potential energies. For more details, we refer the reader to Arbogast, Douglas, and Hornung* [5] *(see also Allaire* [3]*). In the same way, we also have taken a special scaling factor of the Biot-Willis parameters in (2.12)leading to a balance in compressibility/dilation between the matrix and inclusions.*

To set the mathematical framework of our Porblem, we need to introduce the following spaces:

$$\begin{aligned} &\mathbf{H} = H_0^1(\Omega)^3, \qquad L^\varepsilon = L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon), \\ &\mathcal{E}_1^\varepsilon = \left\{ q \in H^1(\Omega_1^\varepsilon);\ q_{|\Gamma} = 0 \right\},\ \mathcal{E}_2^\varepsilon = H^1(\Omega_2^\varepsilon),\ \mathcal{E}^\varepsilon = \mathcal{E}_1^\varepsilon \times \mathcal{E}_2^\varepsilon. \end{aligned}$$

The space $\mathbf{H}$ is equipped with the standard norm: $||\mathbf{v}||_{\mathbf{H}} = ||\mathrm{e}(\mathbf{v})||_{L^2(\Omega)^{3\times 3}}$ and $\mathcal{E}^\varepsilon$ with

$$\|(q_1, q_2)\|_{\mathcal{E}^\varepsilon}^2 = \|\nabla q_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \varepsilon^2 \|\nabla q_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon \|q_1 - q_2\|_{L^2(\Gamma^\varepsilon)}^2 .$$

See S. Monsurro [16]. For a.e. $(t, x) \in Q$, we denote

$$\begin{aligned} \mathbf{u}^\varepsilon(t,x) &= \chi_1^\varepsilon(x)\,\mathbf{u}_1^\varepsilon(t,x) + \chi_2^\varepsilon(x)\,\mathbf{u}_2^\varepsilon(t,x), \\ \mathbb{A}^\varepsilon(x) &= \chi_1^\varepsilon(x)\,\mathbb{A}_1^\varepsilon(x) + \chi_2^\varepsilon(x)\,\mathbb{A}_2^\varepsilon(x), \\ \mathbf{f}^\varepsilon(x) &= \chi_1^\varepsilon(x)\,\mathbf{f}_1(x) + \chi_2^\varepsilon(x)\,\mathbf{f}_2(x). \end{aligned}$$

Note that, thanks to the transmission condition (2.6), the displacement $\mathbf{u}^\varepsilon(t,\cdot)$ lies in $\mathbf{H}$ for a.e. $t \in (0,T)$.

Throughout the paper the following notation will be used: if $\mathcal{F}$ is any functional space then $L_T^p(\mathcal{F})$ denotes the Bochner vector-valued functions space defined by $L_T^p(\mathcal{F}) = L^p(0,T;\mathcal{F})$

The weak formulation of (2.4)-(2.10) can be read as follows**:**

Find $(\mathbf{u}^\varepsilon, p^\varepsilon) \in L_T^\infty(\mathbf{H}) \times L_T^2(\mathcal{E}^\varepsilon)$, such that $p^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon) \in L_T^\infty(L^\varepsilon)$,

$$\partial_t\left(c_1^\varepsilon p_1^\varepsilon + \alpha_1 \mathrm{div}\mathbf{u}^\varepsilon\right) \in L_T^2(\mathcal{E}_1^{\varepsilon *}),\ \partial_t\left(c_2^\varepsilon p_2^\varepsilon + \varepsilon\alpha_2 \mathrm{div}\mathbf{u}^\varepsilon\right) \in L_T^2(\mathcal{E}_2^{\varepsilon *})$$

and for all $\mathbf{v} \in \mathbf{H}$, $(q_1, q_2) \in \mathcal{E}^\varepsilon$, we have

$$\int_\Omega \mathbb{A}^\varepsilon \mathrm{e}(\mathbf{u}^\varepsilon)\,\mathrm{e}(\mathbf{v})\,dx + \int_{\Omega_1^\varepsilon} \alpha_1 \nabla p_1^\varepsilon \mathbf{v} dx + \int_{\Omega_2^\varepsilon} \alpha_2^\varepsilon \nabla p_2^\varepsilon \mathbf{v} dx = \int_\Omega \mathbf{f}^\varepsilon \mathbf{v} dx, \tag{2.13}$$

$$\begin{aligned} &\langle \partial_t\left(c_1^\varepsilon p_1^\varepsilon + \alpha_1 \mathrm{div}\mathbf{u}^\varepsilon\right), q_1\rangle_{\mathcal{E}_1^{\varepsilon *}, \mathcal{E}_1^\varepsilon} + \int_{\Omega_1^\varepsilon} K_1^\varepsilon \nabla p_1^\varepsilon \nabla q_1 dx + \\ &\langle \partial_t\left(c_2^\varepsilon p_2^\varepsilon + \varepsilon\alpha_2 \mathrm{div}\mathbf{u}^\varepsilon\right), q_2\rangle_{\mathcal{E}_2^{\varepsilon *}, \mathcal{E}_2^\varepsilon} + \int_{\Omega_2^\varepsilon} K_2^\varepsilon \nabla p_2^\varepsilon \nabla q_2 dx + \\ &\int_{\Gamma^\varepsilon} g^\varepsilon\left(p_1^\varepsilon - p_2^\varepsilon\right)(q_1 - q_2)\,ds^\varepsilon(x) = 0, \qquad (2.14) \\ &\mathbf{u}^\varepsilon(0,\cdot) = \mathbf{0},\ \chi_1(\cdot)\,p_1^\varepsilon(0,\cdot) + \chi_2(\cdot)\,p_2^\varepsilon(0,\cdot) = 0 \text{ a.e. in } \Omega. \qquad (2.15) \end{aligned}$$

Here and throughout this paper dx and $ds^\varepsilon(x)$ stand respectively for the Lebesgue measure on $\mathbb{R}^3$ and the Hausdorff measure on Γ^ε.

Using assumptions (H1)-(H5), we can establish the following existence and uniqueness result whose proof is a slight modification of that given by R. E. Showalter and B. Momken [18] and therefore will be omitted.

Theorem 2.2. *Assume that (H1)-(H5) hold. Then, for any sufficiently small $\varepsilon > 0$ and $\mathbf{f}^\varepsilon \in \mathbf{L}^2(\Omega)$, there exists a unique couple $(\mathbf{u}^\varepsilon, p^\varepsilon) \in L_T^\infty(\mathbf{H}) \times L_T^2(\mathcal{E}^\varepsilon)$, solution of the weak system (2.13)-(2.15), such that*

$$\|\mathbf{u}^\varepsilon\|_{L_T^\infty(\mathbf{H})} + \|p^\varepsilon\|_{L_T^2(\mathcal{E}^\varepsilon)} + \|p^\varepsilon\|_{L_T^\infty(L^\varepsilon)} \leq C. \tag{2.16}$$

Now, thanks to the a priori estimates (2.16), one is led to study the limiting behavior of the sequence $(\mathbf{u}^\varepsilon, p^\varepsilon)$ as ε approaches 0. To do this, we shall use the two-scale convergence technique that we shall recall hereafter.

First, we define $C_\#(Y)$ to be the space of all continuous functions on $\mathbb{R}^3$ which are Y-periodic. Let the space $L^2_\#(Y)$ (resp. $L^2_\#(Y_i)$, $i = 1, 2$) to be all functions belonging to $L^2_{\text{loc}}(\mathbb{R}^3)$ (resp. $L^2_{\text{loc}}(Z_i)$) which are Y-periodic, and $H^1_\#(Y)$ (resp. $H^1_\#(Y_i)$) to be the space of those functions together with their derivatives belonging to $L^2_\#(Y)$ (resp. $L^2_\#(Z_i)$).

Now, we recall the definition and main results concerning the method of two-scale convergence. For more details, we refer the reader to [3, 4, 17].

Definition 2.3. *A sequence (v^ε) in $L^2(\Omega)$ two-scale converges to $v \in L^2(\Omega \times Y)$ (we write $v^\varepsilon \overset{2-s}{\rightharpoonup} v$) if, for any admissible test function $\varphi \in L^2(\Omega; \mathcal{C}_\#(Y))$,*

$$\lim_{\varepsilon \to 0} \int_\Omega v^\varepsilon(x)\, \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} v(x, y)\, \varphi(x, y)\, dxdy.$$

Theorem 2.4. *Let (v^ε) be a sequence of functions in $L^2(\Omega)$ which is uniformly bounded. Then, there exist $v \in L^2(\Omega \times Y)$ and a subsequence of (v^ε) which two-scale converges to v.*

Theorem 2.5. *Let (v^ε) be a uniformly bounded sequence in $H^1(\Omega)$ (resp. $H^1_0(\Omega)$). Then there exist $v \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$) and $\hat{v} \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that, up to a subsequence,*

$$v^\varepsilon \overset{2-s}{\rightharpoonup} v; \qquad \nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \nabla v + \nabla_y \hat{v}.$$

Here and in the sequel the subscript y on a differential operator denotes the derivative with respect to y.

Theorem 2.6. *Let (v^ε) be a sequence of functions in $H^1(\Omega)$ such that*

$$\|v^\varepsilon\|_{L^2(\Omega)} + \varepsilon \|\nabla v^\varepsilon\|_{L^2(\Omega)^3} \leq C.$$

Then, there exist $v \in L^2\left(\Omega; H^1_\#(Y)\right)$ and a subsequence of (v^ε), still denoted by (v^ε) such that

$$v^\varepsilon \overset{2-s}{\rightharpoonup} v, \qquad \varepsilon \nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \nabla_y v$$

and for every $\varphi \in \mathcal{D}(\Omega; \mathcal{C}_\#(Y))$, we have:

$$\lim_{\varepsilon \to 0} \int_{\Gamma^\varepsilon} v^\varepsilon(x)\, \varphi\left(x, \frac{x}{\varepsilon}\right) ds^\varepsilon(x) = \int_{\Omega \times \Gamma} v(x, y)\, \varphi(x, y)\, dxds(y).$$

Here and in the sequel $ds(y)$ denotes the Hausdorff measure on Γ.

The notion of two-scale convergence can easily be generalized to time-dependent functions without affecting the results stated above. According to [10], we give the following:

Definition 2.7. *We say that a sequence* (v^ε) *in* $L^2(Q)$ *two-scale converges to* $v \in L^2(Q \times Y)$ *(we always write* $v^\varepsilon \overset{2-s}{\rightharpoonup} v$*) if, for any* $\varphi \in L^2(Q; \mathcal{C}_\#(Y))$*:*

$$\lim_{\varepsilon \to 0} \int_Q v^\varepsilon(t,x)\, \varphi\left(t, x, \frac{x}{\varepsilon}\right) dtdx = \int_{Q \times Y} v(t,x,y)\, \varphi(t,x,y)\, dtdxdy.$$

Remark 2.8. *The results stated above still hold for the case of time-dependent sequences. For if* (v^ε) *is a uniformly bounded sequence in* $L^2(Q)$*, there exists then* $v \in L^2(Q)$ *such that, up to a subsequence,* $v^\varepsilon \overset{2-s}{\rightharpoonup} v$ *in the sense of Def. 2.7. Moreover, if* (v^ε) *is uniformly bounded in* $L^2_T(H^1(\Omega))$*, then up to a subsequence, there exist* $v \in L^2_T(H^1(\Omega))$ *and* $v_0 \in L^2\left(Q; H^1_\#(Y)/\mathbb{R}\right)$ *such that* $v^\varepsilon \overset{2-s}{\rightharpoonup} v$ *and* $\nabla v^\varepsilon \overset{2-s}{\rightharpoonup} \nabla v + \nabla_y v_0$*. On the other hand, if a sequence* (v^ε) *is such that*

$$\|v^\varepsilon\|_{L^2(Q)} + \varepsilon \|\nabla v^\varepsilon\|_{L^2(Q)} \leq C,$$

then, up to a subsequence, there exists $v \in L^2_T\left(H^1_\#(Y)\right)$ *such that* $v^\varepsilon \overset{2-s}{\rightharpoonup} v$ *and* $\varepsilon \nabla_y v^\varepsilon \overset{2-s}{\rightharpoonup} \nabla_y v$*.*

In order to state the main result, we shall give in the sequel some notations. Let us first introduce the three following auxilliary problems. For $j,k \in \{1,2,3\}$, let $\mathbf{w}^{jk} \in \left(H^1_\#(Y)/\mathbb{R}\right)^3$ be the solution to the following microscopic system:

$$\begin{cases} -\mathrm{div}_y\left(\mathbb{A}_1 \mathrm{e}_y\left(\mathbf{w}^{jk} + \mathbf{d}^{jk}\right)\right) = 0 \text{ a.e. in } Y_1, \\ -\mathrm{div}_y\left(\mathbb{A}_2 \mathrm{e}_y\left(\mathbf{w}^{jk} + \mathbf{d}^{jk}\right)\right) = 0 \text{ a.e. in } Y_2, \\ \mathbb{A}_1 \mathrm{e}_y\left(\mathbf{w}^{jk} + \mathbf{d}^{jk}\right) \cdot \mathbf{n} = \mathbb{A}_2 \mathrm{e}_y\left(\mathbf{w}^{jk} + \mathbf{d}^{jk}\right) \cdot \mathbf{n} \text{ a.e. on } \Gamma, \\ y \longmapsto \mathbf{w}^{jk}\ Y\text{-periodic} \end{cases}$$

where $\mathbf{d}^{jk}(y) = (y_j \delta_{lk})_{1 \leq l \leq 3}$ and (δ_{kj}) is the Krönecker symbol. For $j = 1,2,3$, let $\pi_j \in H^1(Y_1)/\mathbb{R}$ be the solution of the following stationary micro-pressure equation:

$$\begin{cases} -\mathrm{div}_y\left(K_1\left(\nabla \pi_j + e_j\right)\right) = 0 \text{ in } Y_1, \\ K_1\left(\nabla \pi_j + e_j\right) \cdot \mathbf{n} = 0 \text{ on } \Gamma, \\ y \longmapsto \pi_j\ Y\text{-periodic} \end{cases}$$

where e_j is the j^{th} vector of the canonical basis of $\mathbb{R}^3$. Let $\zeta \in L^\infty_T\left(H^1_\#(Y_2)\right)$ be the unique solution to the following non micro-pressure problem of the Robin type:

$$\begin{cases} \partial_t(c_2 \zeta) - \mathrm{div}_y(K_2 \nabla_y \zeta) = 0 \text{ a.e.in } (0,T) \times Y_2, \\ K_2 \nabla_y \zeta \cdot \mathbf{n} = -g(y)[1 - \zeta] \text{ a.e. on } \Sigma, \\ y \longmapsto \zeta\ Y\text{-periodic}, \\ \zeta(0,y) = 0, \text{ a.e. } y \in Y_2. \end{cases}$$

Now, let us define the homogenized fourth rank tensor $\widetilde{\mathbb{A}} = (\tilde{a}_{j_1 j_2 j_3 j_4})_{1 \leq j_1, j_2, j_3, j_4 \leq 3}$, where the coefficients are given by

$$\tilde{a}_{j_1 j_2 j_3 j_4} = \sum_{k_1, k_2 = 1}^{3} \int_Y a_{j_1 j_2 k_1 k_2}(y)\left(\delta_{j_1 k_1} \delta_{j_2 k_2} + \mathrm{e}_{k_1 k_2, y}\left(\mathbf{w}^{j_3 j_4}\right)(y)\right) dy.$$

Here (a_{jklm}) are the coefficients of the elasticity tensor $\mathbb{A}$ which are given by

$$\mathbb{A}(y) = \chi_1(y)\,\mathbb{A}_1(y) + \chi_2(y)\,\mathbb{A}_2(y) \tag{2.17}$$

for a.e. $y \in Y$, and $\mathrm{e}_{jk,y}(\cdot)$ is the linearized elasticity strain tensor where the derivatives are taken with respect to the microscopic variable y. Let also define the following homogenized tensors:

$$\tilde{\sigma}(\mathbf{u}) = (\tilde{\sigma}_{jk}(\mathbf{u}))\,,\ \tilde{K} = \left(\tilde{K}_{jk}\right),\ B = (b_{jk})\,,\ \Lambda = (\lambda_{jk}) \tag{2.18}$$

where for $j,k \in \{1,2,3\}$

$$\tilde{\sigma}_{jk}(\mathbf{u}) = \sum_{l,m=1}^{3} \tilde{a}_{jklm}\mathrm{e}_{lm}(\mathbf{u}), \tag{2.19}$$

$$\tilde{K}_{jk} = \int_{Y_1} K_1(y)\left(\nabla_y \pi_j + e_j\right)\left(\nabla \pi_k + e_k\right) dy, \tag{2.20}$$

$$b_{jk} = \alpha_1\left(|Y_1|\,\delta_{jk} + \int_\Gamma \pi_k(y)\, n_j ds(y)\right), \tag{2.21}$$

$$\lambda_{jk} = \alpha_1 \int_{Y_1} \sum_{l=1}^{3}\left(\delta_{jl}\delta_{kl} + \frac{\partial w_l^{jk}}{\partial y_l}\right) dy. \tag{2.22}$$

Here $|Y_i|$ denotes the volume of Y_i and $\left(w_l^{ij}\right)_{1\le l\le 3}$ are the components of $\mathbf{w}^{ij}$. Finally let us define the following averaging quantities

$$\mathbf{f} = |Y_1|\mathbf{f}_1 + |Y_2|\mathbf{f}_2, \tag{2.23}$$

$$\tilde{c} = \int_{Y_1} c_1(y)\, dy, \tag{2.24}$$

$$\tilde{g} = \int_\Gamma g(y)\, ds(y) \tag{2.25}$$

and the time-dependent functions

$$\theta(t,\tau) = \alpha_2 \int_\Gamma \partial_t \zeta(t-\tau, y)\, \mathbf{n} ds(y)\,, \tag{2.26}$$

$$\eta(t,\tau) = -\int_\Gamma g(y)\, \partial_t \zeta(t-\tau, y)\, ds(y)\,. \tag{2.27}$$

With these notations, we are now ready to give the main result of the paper:

Theorem 2.9. *Let $(\mathbf{u}^\varepsilon, p^\varepsilon) \in L^\infty(0,T;\mathbf{H}) \times L^2(0,T;\mathcal{E}^\varepsilon)$ be the solution of the weak system (2.13). Then, up to a subsequence, there exists $(\mathbf{u},p) \in L^2\left(0,T;\mathbf{H}_0^1(\Omega)\times H_0^1(\Omega)\right)$ such that*

$$\begin{aligned} \mathbf{u}^\varepsilon &\rightharpoonup \mathbf{u} \text{ in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ weakly,}\\ p_1^\varepsilon &\rightharpoonup p_1 \text{ in } L^2(Q) \text{ weakly,}\\ p_2^\varepsilon &\rightharpoonup \int_{Y_2} p_2(y)\, dy \text{ in } L^2(Q) \text{ weakly,}\end{aligned}$$

where $p = \left(p_1, \int_{Y_2} p_2(y)\, dy\right)$,

$$p_2(t,x,y) = \int_0^t p_1(\tau,x)\, \partial_t\zeta(t-\tau,y)\, d\tau, \text{ a.e. } (t,x,y) \in Q\times Y_2.$$

and the couple $(\mathbf{u}, p_1)$ *is a solution to the homogenized model:*

$$-\mathrm{div}\tilde{\sigma}(\mathbf{u}) + B\nabla p_1 + \int_0^t \theta(t,\tau)\, p_1(\tau, x)\, d\tau = \mathbf{f},\ a.e.\ in\ Q,$$

$$\partial_t\left(\tilde{c}p_1 + \Lambda : \mathrm{e}(\mathbf{u})\right) - \mathrm{div}\left(\tilde{K}\nabla p_1\right) + \tilde{g}p_1 - \int_0^t \eta(t,\tau)\, p_1(\tau, x)\, d\tau = 0,\ a.e.\ in\ Q,$$

$$\mathbf{u} = 0\ and\ \tilde{K}\nabla p_1 \cdot \nu = 0\ a.e.\ on\ \Sigma,$$

$$\mathbf{u}(0,x) = \mathbf{0}\ a.e.\ in\ \Omega,\ p_1(0,x) = 0\ a.e.\ in\ \Omega,$$

Here $\tilde{\sigma}$, B, θ, $\mathbf{f}$, $\tilde{c}$, Λ, $\tilde{K}$, $\tilde{g}$ *and* η *are given in (2.18)-(2.27).*

3. Proof of the main result

As a direct application of the theorems listed above (Thms 2.4-2.6) and the a priori estimates (2.16), we give without proof the following two-scale convergence result concerning the solutions $(\mathbf{u}^\varepsilon, p^\varepsilon)$ of the Problem (2.13)-(2.15).

Theorem 3.1. *There exists a subsequence of* $(\mathbf{u}^\varepsilon, p^\varepsilon)$, *solution of (2.13)-(2.15), still denoted* $(\mathbf{u}^\varepsilon, p^\varepsilon)$, *and there exist*

$$\mathbf{u} \in L_T^\infty(\mathbf{H}), \quad \hat{\mathbf{u}} \in L_T^\infty\left(L^2\left(\Omega; H_\#^1(Y)/\mathbb{R}\right)\right)^3$$

$$p_1 \in L_T^\infty\left(H_0^1(\Omega)\right), \quad \hat{p}_1 \in L^2(Q; H_\#^1(Y)/\mathbb{R})$$

and

$$p_2 \in L_T^\infty\left(L^2\left(\Omega; H_\#^1(Y)\right)\right)$$

such that, for a.e. $t \in (0,T)$,

$$\mathbf{u}^\varepsilon(t,\cdot) \overset{2-s}{\rightharpoonup} \mathbf{u}(t,\cdot), \tag{3.1}$$

$$\chi_1^\varepsilon p_1^\varepsilon(t,\cdot) \overset{2-s}{\rightharpoonup} \chi_1 p_1(t,\cdot), \tag{3.2}$$

$$\chi_1^\varepsilon p_2^\varepsilon(t,\cdot) \overset{2-s}{\rightharpoonup} \chi_2 p_2(t,\cdot) \tag{3.3}$$

in the sense of Def. 2.3 and

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial \mathbf{x}_j} \overset{2-s}{\rightharpoonup} \frac{\partial \mathbf{u}}{\partial \mathbf{x}_j} + \frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{y}_j}, \quad j = 1,2,3, \tag{3.4}$$

$$\chi_1^\varepsilon \nabla p_1^\varepsilon \overset{2-s}{\rightharpoonup} \chi_1\left(\nabla p_1 + \nabla_y \hat{p}_1\right), \tag{3.5}$$

$$\varepsilon\chi_2^\varepsilon \nabla p_2^\varepsilon \overset{2-s}{\rightharpoonup} \chi_2 \nabla_y p_2 \tag{3.6}$$

in the sense of Def. 2.7. Moreover, the following convergence holds:

$$\lim_{\varepsilon\to 0}\int_{\Sigma^\varepsilon} \varepsilon\left(p_1^\varepsilon - p_2^\varepsilon\right)\psi^\varepsilon dt ds^\varepsilon = \int_{Q\times\Gamma}\left(p_1 - p_2\right)\psi dt dx ds, \tag{3.7}$$

for any $\psi \in \mathcal{D}(Q; \mathcal{C}_\#(Y))$ *with* $\psi^\varepsilon(t,x) = \psi(t,x,x/\varepsilon)$.

To determine the limiting equations of the system (2.13)-(2.15), we begin by choosing the adequate admissible test functions. Let $\mathbf{v}^\varepsilon(x) = \mathbf{v}(x) + \varepsilon\hat{\mathbf{v}}\left(x, \frac{x}{\varepsilon}\right)$ where $\mathbf{v} \in \mathcal{D}(\Omega)^3$ and $\hat{\mathbf{v}} \in \mathcal{D}\left(\Omega; \mathcal{C}_\#^\infty(Y)\right)^3$. Let also $q_1^\varepsilon(t,x) = \varphi_1(t,x) + \varepsilon\hat{\varphi}_1\left(t, x, \frac{x}{\varepsilon}\right)$

and $q_2^\varepsilon(t,x) = \varphi_2\left(t,x,\dfrac{x}{\varepsilon}\right)$ where $\varphi_1 \in \mathcal{D}\left((0,T)\times\bar{\Omega}\right)$ and $\varphi_2, \hat{\varphi}_1 \in \mathcal{D}\left(Q;\mathcal{C}_\#^\infty(Y)\right)$. Taking $\mathbf{v} = \mathbf{v}^\varepsilon$ in (2.13), we have:

$$
\begin{aligned}
\int_\Omega \mathbf{f}^\varepsilon \mathbf{v}^\varepsilon dx &= \int_\Omega \mathbb{A}^\varepsilon(x)\,\mathrm{e}(\mathbf{u}^\varepsilon)\,\mathrm{e}(\mathbf{v}^\varepsilon)\,dx + \int_{\Omega_1^\varepsilon} \alpha_1 \nabla p_1^\varepsilon \mathbf{v}^\varepsilon dx + \varepsilon\int_{\Omega_2^\varepsilon} \alpha_2 \nabla p_2^\varepsilon \mathbf{v}^\varepsilon dx \\
&= \int_\Omega \mathbb{A}^\varepsilon(x)\,\mathrm{e}(\mathbf{u}^\varepsilon)\left(\mathrm{e}(\mathbf{v})(x) + \mathrm{e}_y(\hat{\mathbf{v}})\left(x,\frac{x}{\varepsilon}\right)\right)dx + \\
&\quad \int_\Omega \left(\alpha_1\chi_1^\varepsilon(x)\nabla p_1^\varepsilon + \varepsilon\alpha_2\chi_2^\varepsilon(x)\nabla p_2^\varepsilon\right)\mathbf{v}(x)\,dx + \varepsilon R_1^\varepsilon, \qquad (3.8)
\end{aligned}
$$

where

$$
\begin{aligned}
R_1^\varepsilon &= \int_\Omega \mathbb{A}^\varepsilon(x)\,\mathrm{e}(\mathbf{u}^\varepsilon)\,\mathrm{e}_x(\mathbf{w})\left(x,\frac{x}{\varepsilon}\right)dx + \alpha_1\int_\Omega \chi_1^\varepsilon(x)\nabla p_1^\varepsilon \mathbf{w}\left(x,\frac{x}{\varepsilon}\right)dx \\
&\quad + \varepsilon\alpha_2\int_\Omega \chi_2^\varepsilon(x)\nabla p_2^\varepsilon \mathbf{w}\left(x,\frac{x}{\varepsilon}\right)dx.
\end{aligned}
$$

Observe that $R_1^\varepsilon = O(1)$.

Now, we pass to the limit in (3.8). In view of (3.4), and since $\mathbb{A}^t(\mathrm{e}(\mathbf{v}) + \mathrm{e}_y(\hat{\mathbf{v}}))$ is an admissible test function, the first integral in the l.h.s. of (3.8) converges to

$$
\int_{\Omega\times Y} \mathbb{A}\left(\mathrm{e}(\mathbf{u}) + \mathrm{e}_y(\hat{\mathbf{u}})\right)\left(\mathrm{e}(\mathbf{v}) + \mathrm{e}_y(\hat{\mathbf{v}})\right)dxdy \qquad (3.9)
$$

where the tensor $\mathbb{A}(y)$ is given by (2.17). In view of Divergence Lemma and (3.5)-(3.6), the second integral of the l.h.s. of (3.8) tends to

$$
\begin{aligned}
&\alpha_1\int_{\Omega\times Y_1}\left(\nabla p_1 + \nabla_y\hat{p}_1\right)\mathbf{v}(x)\,dxdy + \alpha_2\int_{\Omega\times Y_2}\nabla_y p_2 \mathbf{v}(x)\,dxdy \\
= \;& \alpha_1|Y_1|\int_\Omega \nabla p_1 \mathbf{v}(x)\,dx + \int_{\Omega\times\Gamma}\left(\alpha_1\hat{p}_1 + \alpha_2 p_2\right)(\mathbf{v}\cdot\mathbf{n})\,dxds, \qquad (3.10)
\end{aligned}
$$

By Theorem 2.4, it follows that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega \mathbf{f}^\varepsilon\mathbf{v}^\varepsilon(x)\,dx &= \lim_{\varepsilon\to 0}\left(\int_\Omega \mathbf{f}^\varepsilon(x)\,\mathbf{v}(x)\,dx + \varepsilon\int_\Omega \mathbf{f}^\varepsilon(x)\,\hat{\mathbf{v}}\left(x,\frac{x}{\varepsilon}\right)dx\right) \\
&= \int_\Omega \mathbf{f}\mathbf{v}(x)\,dx \qquad (3.11)
\end{aligned}
$$

where $\mathbf{f}$ is given by (2.23). Thus, collecting these limits (3.9)-(3.11), we obtain the limiting equation of (3.8)

$$
\begin{aligned}
&\int_{\Omega\times Y}\mathbb{A}\left[\mathrm{e}(\mathbf{u}) + \mathrm{e}_y(\hat{\mathbf{u}})\right]\left[\mathrm{e}(\mathbf{v}) + \mathrm{e}_y(\hat{\mathbf{v}})\right]dxdy + \alpha_1|Y_1|\int_\Omega \nabla p_1\mathbf{v}dx \\
&+ \int_{\Omega\times\Gamma}\left(\alpha_1\hat{p}_1 + \alpha_2 p_2\right)(\mathbf{v}\cdot\mathbf{n})\,dxds = \int_\Omega \mathbf{f}\mathbf{v}dx \qquad (3.12)
\end{aligned}
$$

which is valid for a.e. $t \in (0,T)$. Next, we proceed to get the limiting equation of (2.14). Taking $q_1 = q_1^\varepsilon$ and $q_2 = q_2^\varepsilon$ in (2.14), integrating by parts over $(0,T)$ and taking into account the initial conditions (2.15), we obtain

$$
\begin{aligned}
&-\int_{Q_1^\varepsilon}\left(c_1^\varepsilon(x)\,p_1^\varepsilon + \alpha_1\mathrm{div}\mathbf{u}^\varepsilon\right)\partial_t\varphi_1(t,x)\,dtdx - \int_{Q_2^\varepsilon} c_2^\varepsilon(x)\,p_2^\varepsilon\partial_t\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx + \\
&\int_{Q_1^\varepsilon} K_1\left(\frac{x}{\varepsilon}\right)\nabla p_1^\varepsilon\left(\nabla\varphi_1(t,x) + \nabla_y\hat{\varphi}_1\left(t,x,\frac{x}{\varepsilon}\right)\right)dtdx +
\end{aligned}
$$

$$\int_{Q_2^\varepsilon}\varepsilon k_2\left(\frac{x}{\varepsilon}\right)\nabla p_2^\varepsilon\nabla_y\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx+$$

$$\varepsilon\int_{\Sigma^\varepsilon}g\left(\frac{x}{\varepsilon}\right)(p_1^\varepsilon-p_2^\varepsilon)\left(\varphi_1(t,x)-\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)\right)dtds^\varepsilon+\varepsilon R_2^\varepsilon \tag{3.13}$$

where

$$R_2^\varepsilon=\int_{Q_1^\varepsilon}-\left(c_1^\varepsilon(x)\,p_1^\varepsilon+\alpha_1\mathrm{div}\mathbf{u}^\varepsilon\right)\partial_t\hat{\varphi}_1\left(t,x,\frac{x}{\varepsilon}\right)dtdx+$$
$$\int_{Q_2^\varepsilon}-\alpha_2\mathrm{div}\mathbf{u}^\varepsilon\partial_t\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx+$$
$$\int_{Q_1^\varepsilon}K_1\left(\frac{x}{\varepsilon}\right)\nabla p_1^\varepsilon\nabla_x\hat{\varphi}_1\left(t,x,\frac{x}{\varepsilon}\right)dtdx+$$
$$\varepsilon\int_{Q_1^\varepsilon}K_2\left(\frac{x}{\varepsilon}\right)\nabla p_2^\varepsilon\nabla_x\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx+$$
$$\varepsilon\int_{\Sigma^\varepsilon}g\left(\frac{x}{\varepsilon}\right)(p_1^\varepsilon-p_2^\varepsilon)\,\hat{\varphi}_1(t,x)\,dtds^\varepsilon.$$

The first integral of (3.13) is equal to

$$\int_{\Omega_T}-\chi_1\left(\frac{x}{\varepsilon}\right)\left(c_1\left(\frac{x}{\varepsilon}\right)p_1^\varepsilon+\alpha_1\mathrm{div}\mathbf{u}^\varepsilon\right)\partial_t\varphi_1(t,x)\,dtdx,$$

and thanks to (3.2) and (3.4), converges to

$$\int_{Q\times Y}-\chi_1(y)\left(c_1(y)\,p_1+\alpha_1\left(\mathrm{div}\mathbf{u}+\mathrm{div}_y\hat{\mathbf{u}}\right)\right)\partial_t\varphi_1(t,x)\,dtdxdy.$$

In a similar way, by (3.3) and (3.4), it follows that

$$\int_{Q_2^\varepsilon}c_2^\varepsilon(x)\,p_2^\varepsilon\partial_t\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx\to\int_{Q\times Y}\chi_2(y)\,c_2(y)\,p_2\partial_t\varphi_2(t,x,y)\,dtdxdy$$

Now, in view of (3.5) one can deduce that

$$\int_{Q_1^\varepsilon}K_1\left(\frac{x}{\varepsilon}\right)\nabla p_1^\varepsilon\left(\nabla\varphi_1(t,x)+\nabla_y\hat{\varphi}_1\left(t,x,\frac{x}{\varepsilon}\right)\right)dtdx=$$
$$\int_Q\chi_1\left(\frac{x}{\varepsilon}\right)K_1\left(\frac{x}{\varepsilon}\right)\nabla p_1^\varepsilon\left(\nabla\varphi_1(t,x)+\nabla_y\hat{\varphi}_1\left(t,x,\frac{x}{\varepsilon}\right)\right)dtdx\to$$
$$\int_{Q\times Y}\chi_1(y)\,K_1(y)\left(\nabla p_1+\nabla_y\hat{p}_1\right)\left(\nabla\varphi(t,x)+\nabla_y\hat{\varphi}_1(t,x,y)\right)dtdxdy$$

and thanks to (3.6), we also get

$$\begin{aligned}&\int_{Q_2^\varepsilon}\varepsilon k_2\left(\frac{x}{\varepsilon}\right)\nabla p_2^\varepsilon\nabla_y\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx\\ =\;&\int_Q\chi_2\left(\frac{x}{\varepsilon}\right)K_2\left(\frac{x}{\varepsilon}\right)\varepsilon\nabla p_2^\varepsilon\nabla_y\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)dtdx\to\\ &\int_{Q\times Y}\chi_2(y)\,K_2(y)\,\nabla p_2\nabla_y\varphi_2(t,x,y)\,dtdxdy.\end{aligned}$$

By virtue of (3.7), we find that

$$\varepsilon\int_{\Sigma^\varepsilon}g\left(\frac{x}{\varepsilon}\right)(p_1^\varepsilon-p_2^\varepsilon)\left(\varphi_1(t,x)-\varphi_2\left(t,x,\frac{x}{\varepsilon}\right)\right)dtds^\varepsilon\to$$

$$\int_{Q\times\Gamma} g\left(y\right)\left(p_1-p_2\right)\left(\varphi_1\left(t,x\right)-\varphi_2\left(t,x,y\right)\right)dtdsdy.$$

As before, we observe that $R_2^{\varepsilon}=O\left(1\right)$ and, by collecting all the preceeding limits, we get the following limiting equation of (2.14) :

$$\begin{aligned}&\int_{Q\times Y_1}-\left(c_1\left(y\right)p_1+\alpha_1\left(\mathrm{div}\mathbf{u}+\mathrm{div}_y\hat{\mathbf{u}}\right)\right)\partial_t\varphi_1 dtdxdy+\\&\int_{Q\times Y_1}K_1\left(y\right)\left(\nabla p_1+\nabla_y\hat{p}_1\right)\left(\nabla\varphi_1+\nabla_y\hat{\varphi}_1\right)dtdxdy+\\&\int_{Q\times Y_2}\left(-c_2\left(y\right)p_2\partial_t\varphi_2+K_2\left(y\right)\nabla_y p_2\nabla_y\varphi_2\right)dtdxdy+\\&\int_{Q\times\Gamma}g\left(y\right)\left(p_1-p_2\right)\left(\varphi_1-\varphi_2\right)dtdsdy=0.\end{aligned}\tag{3.14}$$

By density argument, the equations (3.12) and (3.14) still hold true for any $(\mathbf{v},\hat{\mathbf{v}})\in\mathbf{H}\times L^2\left(\Omega,H^1\left(Y\right)/\mathbb{R}\right)^3$ and $(\varphi_1,\hat{\varphi}_1,\varphi_2)\in L_T^2\left(H^1\left(\Omega\right)\right)\times L^2\left(Q;H_\#^1\left(Y\right)/\mathbb{R}\right)\times L^2\left(Q;H_\#^1\left(Y\right)\right)$. We can summarize the preceding by observing that these equations are a weak formulation associated to the two-scale homogenized system (3.15)-(3.31). Indeed, integrating by parts in (3.12) and (3.14), we obtain the following system:

$$-\mathrm{div}_y\left(\mathbb{A}_1\left[\mathrm{e}\left(\mathbf{u}\right)+\mathrm{e}_y\left(\hat{\mathbf{u}}\right)\right]\right)=0 \text{ a.e.in } Q\times Y_1,\tag{3.15}$$

$$-\mathrm{div}_y\left(\mathbb{A}_2\left[\mathrm{e}\left(\mathbf{u}\right)+\mathrm{e}_y\left(\hat{\mathbf{u}}\right)\right]\right)=0 \text{ a.e.in } Q\times Y_2,\tag{3.16}$$

$$\begin{aligned}&-\mathrm{div}\left(\int_Y\mathbb{A}\left[\mathrm{e}\left(\mathbf{u}\right)+\mathrm{e}_y\left(\hat{\mathbf{u}}\right)\right]dy\right)+\alpha_1\left|Y_1\right|\nabla p_1+\\&\int_\Gamma\left(\alpha_1\hat{p}_1+\alpha_2 p_2\right)\mathbf{n}ds=\mathbf{f}\text{ a.e. in }Q,\end{aligned}\tag{3.17}$$

and

$$-\mathrm{div}_y\left(K_1\left(\nabla p_1+\nabla_y\hat{p}_1\right)\right)=0\text{ a.e. in }Q\times Y_1,\tag{3.18}$$

$$\partial_t\left(c_2p_2\right)-\mathrm{div}_y\left(K_2\nabla_y p_2\right)=0\text{ a.e.in }Q\times Y_2,\tag{3.19}$$

$$\begin{aligned}&\partial_t\left(\int_{Y_1}\left(c_1p_1+\alpha_1\left(\mathrm{div}\mathbf{u}+\mathrm{div}_y\hat{\mathbf{u}}\right)\right)\right)-\mathrm{div}\left(\int_{Y_1}K_1\left(\nabla p_1+\nabla_y\hat{p}_1\right)dy\right)+\\&\int_\Gamma g\left(y\right)\left[p_1-p_2\right]ds\left(y\right)=0\text{ a.e. in }Q,\end{aligned}\tag{3.20}$$

with the transmission and boundary conditions:

$$\mathbb{A}_1\left[\mathrm{e}\left(\mathbf{u}\right)+\mathrm{e}_y\left(\hat{\mathbf{u}}\right)\right]\cdot\mathbf{n}=\mathbb{A}_2\left[\mathrm{e}\left(\mathbf{u}\right)+\mathrm{e}_y\left(\hat{\mathbf{u}}\right)\right]\cdot\mathbf{n}\text{ a.e. on }Q\times\Gamma,\tag{3.21}$$

$$\left(K_1\left(\nabla p_1+\nabla_y\hat{p}_1\right)\right)\cdot\mathbf{n}=0\text{ a.e. on }Q\times\Gamma,\tag{3.22}$$

$$\left(K_1\left(\nabla p_1+\nabla_y \hat{p}_1\right)\right)\cdot v=0 \text{ a.e. on } (0,T)\times\partial\Omega\times Y_1, \tag{3.23}$$
$$K_2\nabla_y p_2\cdot\mathbf{n}= -g(y)\left[p_1-p_2\right] \text{ a.e. on } Q\times\Gamma, \tag{3.24}$$

$$\mathbf{u}=0 \text{ a.e. on } \partial\Omega, \tag{3.25}$$

$$y\longmapsto \hat{\mathbf{u}},\ \hat{p}_1, p_2 \ Y\text{-periodic}, \tag{3.26}$$

and the initial conditions:

$$\mathbf{u}(0,x)=\mathbf{0} \text{ a.e. in } \Omega, \tag{3.27}$$
$$\hat{\mathbf{u}}(0,x,y)=\mathbf{0} \text{ a.e. in } \Omega\times Y, \tag{3.28}$$
$$p_1(0,x)=0 \text{ a.e. in } \Omega, \tag{3.29}$$
$$\hat{p}_1(0,x,y)=0 \text{ a.e. in } \Omega\times Y_1 \tag{3.30}$$
$$p_2(0,x,y)=0 \text{ a.e. in } \Omega\times Y_2. \tag{3.31}$$

Now we decouple the system (3.15)-(3.31). In view of the linearity of the two first equations (3.15)-(3.16), we can write that, up to an additive constant:

$$\hat{\mathbf{u}}(t,x,y)=\sum_{i,j=1}^{3}\mathrm{e}_{ij}(\mathbf{u})(t,x)\,\mathbf{w}^{ij}(y)+C^{te},\ \text{ a.e. } (t,x,y)\in Q\times Y, \tag{3.32}$$

where, for $i,j\in\{1,2,3\}$, $\mathbf{w}^{ij}\in\left(H^1_{\#}(Y)/\mathbb{R}\right)^3$ is the solution to the following microscopic system:

$$-\mathrm{div}_y\left(\mathbb{A}_1\mathrm{e}_y\left(\mathbf{w}^{ij}+\mathbf{d}^{ij}\right)\right)=0 \text{ a.e. in } Y_1,$$

$$-\mathrm{div}_y\left(\mathbb{A}_2\mathrm{e}_y\left(\mathbf{w}^{ij}+\mathbf{d}^{ij}\right)\right)=0 \text{ a.e. in } Y_2,$$

$$\mathbb{A}_1\mathrm{e}_y\left(\mathbf{w}^{ij}+\mathbf{d}^{ij}\right)\cdot\mathbf{n}=\mathbb{A}_2\mathrm{e}_y\left(\mathbf{w}^{ij}+\mathbf{d}^{ij}\right)\cdot\mathbf{n} \text{ a.e. on } \Gamma,$$

$$y\longmapsto \mathbf{w}^{ij}\ Y\text{-periodic}.$$

Here $\mathbf{d}^{kl}=\left(y_K\delta_{il}\right)_{1\le i\le 3}$ and (δ_{ij}) is the Krönecker symbol.

Similarly, in view of (3.18), (3.22) and (3.26) one can write that:

$$\hat{p}_1(t,x,y)=\sum_{i=1}^{3}\frac{\partial p_1}{\partial x_i}(t,x)\,\pi_i(y)+C^{te},\ \text{ a.e. } (t,x,y)\in Q\times Y_1, \tag{3.33}$$

where, for $i=1,2,3$, the micro-pressure $\pi_i\in H^1(Y_1)/\mathbb{R}$ is the solution of the following stationary equation:

$$-\mathrm{div}_y\left(K_1\left(\nabla\pi_i+e_i\right)\right)=0 \text{ in } Y_1,$$
$$K_1\left(\nabla\pi_i+e_i\right)\cdot\mathbf{n}=0 \text{ on } \Gamma,$$
$$y\longmapsto\pi_i\ Y\text{-periodic}.$$

Here e_i is the i^{th} vector of the canonical basis of $\mathbb{R}^3$. Let us denote

$$\widetilde{\mathbb{A}} \;=\; \left(\tilde{a}_{i_1i_2i_3i_4}\right)_{1\le i_1,i_2,i_3,i_4\le 3},$$

$$\tilde{a}_{i_1 i_2 i_3 i_4} = \sum_{j_1,j_2=1}^{3} \int_Y a_{i_1 i_2 j_1 j_2}(y) \left(\delta_{i_1 j_1}\delta_{i_2 j_2} + \mathrm{e}_{j_1 j_2, y}\left(\mathbf{w}^{i_3 i_4}\right)(y)\right) dy,$$

where (a_{ijlm}) are the coefficients of the elasticity tensor $\mathbb{A}$ and

$$\mathrm{e}_{ij,y}(\mathbf{w}) = \frac{1}{2}\left(\frac{\partial w_i}{\partial y_j} + \frac{\partial w_j}{\partial y_i}\right), \ \mathbf{w} = (w_j)_{1\le j\le 3}.$$

Let also define the effective stress tensor

$$\tilde{\sigma}(\mathbf{u}) = (\tilde{\sigma}_{ij}(\mathbf{u}))_{1\le i,j\le 3}, \quad \tilde{\sigma}_{ij}(\mathbf{u}) = \sum_{l,m=1}^{3} \tilde{a}_{ijlm}\mathrm{e}_{lm}(\mathbf{u}),$$

the effective permeability tensor

$$\tilde{K} = \left(\tilde{K}_{ij}\right)_{1\le i,j\le 3}, \quad \tilde{K}_{ij} = \int_{Y_1} K_1(y)\left(\nabla_y \pi_i + e_i\right)\left(\nabla \pi_j + e_j\right) dy,$$

the effective Biot-Willis matrices:

$$\begin{aligned} B &= (b_{ij}), \ b_{ij} = \alpha_1 \left(|Y_1|\,\delta_{ij} + \int_\Gamma \pi_j(y)\, n_i ds(y)\right), \ \mathbf{n} = (n_i)_{1\le i\le 3} \\ \Lambda &= (\lambda_{ij})_{1\le i,j\le 3}, \quad \lambda_{ij} = \alpha_1 \int_{Y_1} \sum_{m=1}^{3}\left(\delta_{im}\delta_{jm} + \frac{\partial w_m^{ij}}{\partial y_m}\right) dy, \\ \mathbf{w}^{ij} &= \left(w_m^{ij}\right)_{1\le m\le 3} \end{aligned}$$

and finally the following averaging quantities

$$\tilde{c} = \int_{Y_1} c_1(y)\, dy, \quad \tilde{g} = \int_\Gamma g(y)\, ds(y).$$

Then from (3.32)-(3.33) we deduce the homogenized system :

$$-\mathrm{div}\tilde{\sigma}(\mathbf{u}) + B\nabla p_1 + \alpha_2 \int_\Gamma p_2 \mathbf{n} ds(y) = \mathbf{f} \text{ a.e. in } Q, \tag{3.34}$$

$$\partial_t\left(\tilde{c}p_1 + \Lambda : \mathrm{e}(\mathbf{u})\right) - \mathrm{div}\left(\tilde{K}\nabla p_1\right) + \tilde{g}p_1 - \int_\Gamma g(y)\, p_2 ds(y) = 0, \text{ a.e. in } Q, \tag{3.35}$$

$$\partial_t(c_2 p_2) - \mathrm{div}_y(K_2 \nabla_y p_2) = 0 \text{ a.e. in } Q\times Y_2, \tag{3.36}$$

$$c_2 \nabla_y p_2 \cdot \mathbf{n} = -g(y)\left[p_1 - p_2\right] \text{ a.e. on } Q\times\Gamma, \tag{3.37}$$

$$\mathbf{u} = 0 \text{ and } \tilde{K}\nabla p_1 \cdot \nu = 0 \text{ a.e. on } (0,T)\times\Sigma, \tag{3.38}$$

$$y \longmapsto p_2 \ Y\text{-periodic}, \tag{3.39}$$

$$\mathbf{u}(0,x) = \mathbf{0} \text{ a.e. in } \Omega, \ p_1(0,x) = 0 \text{ a.e. in } \Omega, \tag{3.40}$$

$$p_2(0,x,y) = 0 \text{ a.e. in } \Omega\times Y_2. \tag{3.41}$$

Now, we are going to establish a relation between the two pressures p_1 and p_2. To this aim, let $\zeta \in L^\infty\left(0,T; H^1_\#(Y_2)\right)$ be the unique solution to the following microscopic and non homogeneous Robin problem:

$$\begin{aligned} &\partial_t(c_2\zeta) - \mathrm{div}_y(K_2\nabla_y\zeta) = 0 \text{ a.e.in } (0,T)\times Y_2, \\ &K_2\nabla_y\zeta\cdot\mathbf{n} = -g(y)\left[1-\zeta\right] \text{ a.e. on } \Sigma, \\ &y \longmapsto \zeta \ Y\text{-periodic}, \end{aligned}$$

$$\zeta(0,y)=0, \text{ a.e. } y\in Y_2.$$

Since c_2, K_2, g are time-independent and p_1 is independent of y, using the Laplace transform method, one can then easily see that

$$p_2(t,x,y)=\int_0^t p_1(\tau,x)\,\partial_t\zeta(t-\tau,y)\,d\tau, \text{ a.e. } (t,x,y)\in Q\times Y_2. \tag{3.42}$$

Therefore, the homogenized system (3.34)-(3.41) can be rewritten as

$$\begin{aligned}
&-\mathrm{div}\tilde{\sigma}(\mathbf{u})+B\nabla p_1+\int_0^t\theta(t,\tau)\,p_1(\tau,x)\,d\tau=\mathbf{f} \text{ a.e. in } Q,\\
&\partial_t(\tilde{c}p_1+\Lambda:\mathrm{e}(\mathbf{u}))-\mathrm{div}\left(\tilde{K}\nabla p_1\right)+\tilde{g}p_1\\
&-\int_0^t\eta(t,\tau)\,p_1(\tau,x)\,d\tau=0, \text{ a.e. in } Q,\\
&\mathbf{u}=0 \text{ and } \tilde{K}\nabla p_1\cdot\nu=0 \text{ a.e. on } (0,T)\times\partial\Omega,\\
&\mathbf{u}(0,x)=\mathbf{0},\ p_1(0,x)=0 \text{ a.e. in } \Omega,
\end{aligned}$$

where we have denoted

$$\begin{aligned}
\theta(t,\tau) &= \alpha_2\int_\Gamma\partial_t\zeta(t-\tau,y)\,\mathbf{n}ds(y),\\
\eta(t,\tau) &= \int_\Gamma g(y)\,\partial_t\zeta(t-\tau,y)\,ds(y).
\end{aligned}$$

Finally, let us observe that the overall pressure of the fluid flow in the microstructure model which is given by

$$P^\varepsilon(t,x)=\chi_1^\varepsilon(x)p_1^\varepsilon(t,x)+\chi_2^\varepsilon(x)p_2^\varepsilon(t,x)$$

for a.e. $(t,x)\in Q$, two-scale converges to $\chi_1(y)p_1(t,x)+\chi_2(y)p_2(t,x,y)$, and thanks to (3.42), converges then weakly in $L^2(Q)$ to

$$|Y_1|p_1(t,x)+\int_0^t\int_{Y_2}p_1(\tau,x)\,\partial_t\zeta(t-\tau,y)\,dyd\tau.$$

This concludes the proof of Theorem 2.9.

4. Conclusion

We have used the homogenization theory to derive a macro-model for fluid flow in composite poroelastic with microstructures, in which inclusions are fully embedded and with very low permeabilities. We have shown that the overall behavior of fluid flow in such heterogeneous media with low permeability at the micro-scale may present memory terms. We also have shown that in such cases, the Biot-Willis parameters are, as in [2], matrices and no longer scalars, as it is usually considered in the poroelasticity litterature, since it is assumed there that the medium is homogeneous and isotropic. Nevertheless, anisotropic media may present different coupling interaction properties in different directions at the micro-scale, and which lead at the macro-scale to such anisotropic Biot-Willis parameters. Finally, let us mention that the result of the paper remains valid if one considers non homogeneous initial conditions or with any volume distributed source densities in each phases.

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